OLIMPIADA MATEMÁTICA

ACTIVIDADES 2021

REAL SOCIEDAD MATEMÁTICA ESPAÑOLA

Continuamos este año la publicación de este pequeño folleto que recoge los problemas propuestos en las principales actividades relacionadas con la Olimpiada Matemática Española: desde los utilizados en las fases local y nacional de su edición número 57, hasta los de las olimpiadas y concursos internacionales en las que los equipos españoles de matemáticas, allí seleccionados, participaron en el año 2021. Con ello pretendemos poner al alcance tanto de profesores como de estudiantes el valioso material de trabajo que suponen los buenos problemas. Esperamos y deseamos que pueda ser utilizado en diferentes actividades de formación, imprescindibles para poder enfrentarse con éxito a cualquier competición matemática; actividades que difícilmente pueden llegar a buen puerto sin la desinteresada y nunca reconocida labor de los profesores de secundaria.

En las páginas que siguen encontraréis sin duda problemas interesantes, atractivos y formativos. Un buen problema puede abordarse desde distintos enfoques, que os invitamos a explorar. Pero un buen problema es sobre todo un reto. Os animamos a aceptarlo, confiando en que disfrutaréis con ello.

La Olimpiada Internacional de Matemáticas (IMO), ha servido de modelo para el resto de competiciones, destinadas a estudiantes de secundaria, que cada año se realizan a lo largo y ancho de los cinco continentes. A diferencia de otras olimpiadas científicas no tiene establecido ningún temario, aunque por tradición los problemas que en ella se proponen, siempre abordables con técnicas elementales, se agrupan alrededor de cuatro grandes bloques: álgebra, geometría, teoría de números y combinatoria. En la web hay excelentes páginas en inglés con abundantes recursos: sirvan como ejemplo las de Art of Problem solving (www.artofproblemsolving.com), y la de Mathlinks (www.mathlinks.ro). Es de destacar asimismo The IMO Compendium (www.imomath.com), que recoge los problemas de las llamadas ”listas cortas”, entre los que cada año se eligen los seis que constituyen finalmente la prueba de la Olimpiada Internacional. No son muchas las páginas en español sobre el tema, aunque en la de la Olimpiada Matemática Argentina (www.oma.org.ar) puede encontrarse también variado material.

Deseamos agradecer de todo corazón el trabajo altruista de cuantos hacen posible, año tras año, la Olimpiada Matemática Española. Estamos especialmente agradecidos a los estudiantes, que cada año participan en ella con ilusión, y a sus profesores, que los alientan y ayudan. La RSME, a través de su Comisión de Olimpiadas, espera ser capaz de ofrecerles el apoyo y soporte matemático que necesiten.

María Gaspar Alonso-Vega

Introducción

Como cada año desde 1964, la Real Sociedad Matemática Española organizó el Concurso Final de la 57.ª Olimpiada Matemática Española (OME). Lo hizo a través de los Delegados de Distrito o Comunidad, y bajo la coordinación de la Comisión de Olimpiadas de la RSME con la colaboración del Ministerio de Educación y Formación Profesional. La OME se desarrolla en dos fases: la primera o Fase Local se celebró a primeros de año y en ella se escogieron a los estudiantes que representaron a cada comunidad autónoma en la Segunda Fase. Este año el Concurso Final, que es de ámbito nacional, se celebró telemáticamente en el mes de mayo y en él se seleccionaron a los miembros del equipo español que han representado a nuestro país en la Olimpiada Internacional (IMO), organizada por Rusia y celebrada telemáticamente en julio con sede en Barcelona, y la Olimpiada Iberoamericana de Matemáticas, celebrada telemáticamente y organizada por Perú en octubre.

Como hacen otros países de nuestro entorno es fundamental que nuestros estudiantes se preparen lo mejor posible, dentro de nuestras posibilidades, para competir en estos concursos. Cada año, el equipo español que acude a la IMO es invitado a participar en Olimpiadas internacionales de ámbito regional previas a la IMO. Como parte de la preparación del equipo español, este año los estudiantes seleccionados en el Concurso Final compartieron unas sesiones de trabajo con exolímpicos en Barcelona antes de participar en la IMO.

Los resultados obtenidos por los equipos que representaron a España en las diferentes competiciones internacionales son una prueba clara del trabajo realizado y de la progresión que se viene mostrando en los últimos años. El calendario olímpico internacional comenzó en el mes de abril con la Olimpiada Matemática Femenina Europea (EGMO), organizada virtualmente por Georgia. En el mes de julio, los resultados de la IMO nos situaron en el puesto 63º: Àlex Rodríguez cosechó una medalla de bronce, mientras que Leonardo Costa, Miguel Navarro y Miguel Valdivieso se hicieron con una mención de honor. En la Olimpiada Iberoamericana el equipo se situó en una meritoria quinta posición, con las medallas de plata de Leonardo Costa y Àlex Rodríguez y los bronces de Bernat Pagès y Miguel Valdivieso.

En este folleto se recogen los enunciados y soluciones de la Fase Local y la Fase Nacional, los enunciados y soluciones del Mathcontest, de la Barcelona Spring Math-Olympiad y los enunciados de la EGMO y la Olimpiada Mediterránea. También se han incluido los enunciados de la IMO. Finalmente, se han incluido los enunciados y soluciones de la Olimpiada Iberoamericana.

Óscar Rivero Salgado y José Luis Díaz Barrero

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Enunciados

1. Determinar todos los números de cuatro cifras $n = \overline{abcd}$ tales que al insertar un dígito 0 en cualquier posición se obtiene un múltiplo de 7.

2. Determinar todas las parejas de enteros positivos (m, n) para los cuales es posible colocar algunas piedras en las casillas de un tablero de m filas y n columnas, no más de una piedra por casilla, de manera que todas las columnas tengan la misma cantidad de piedras, y no existan dos filas con la misma cantidad de piedras.

3. En el triángulo ABC con lado mayor BC, las bisectrices se cortan en I. Las rectas AI, BI, CI cortan a BC, CA, AB en los puntos D, E, F, respectivamente. Se consideran puntos G y H en los segmentos BD y CD, respectivamente, tales que $\angle GID = \angle ABC$ y $\angle HID = \angle ACB$. Probar que $\angle BHE = \angle CGF$.

4. Al desarrollar $(1 + x + x^2)^n$ en potencias de x, exactamente tres términos tienen coeficiente impar. ¿Para qué valores de n es esto posible?

5. En un torneo de ajedrez participan 8 maestros durante 7 días. Cada día se disputan 4 partidas en las cuales participan todos los maestros, y al finalizar el torneo todos se han enfrentado contra todos exactamente una vez. Demostrar que al terminar el quinto día del torneo existe un conjunto de al menos 4 maestros que ya han jugado entre ellos todas las partidas.

6. $ABCD$ es un cuadrilatero convexo verificando $AB > BC$, $CD = DA$ y $\angle ABD = \angle DBC$. Sea E el punto de la recta AB tal que $\angle DEB = 90^circ$. Probar que $AE = \dfrac{AB - BC}{2}$.

7. Demostrar que todos los números racionales pueden expresarse como suma de algunas fracciones de la forma $\dfrac{n-1}{n+2}$, con $n \geq 0$ entero, admitiendo repetir sumandos.

8. Determinar todas las funciones f tales que

$$f(xf(y) + y) = f(xy) + f(y)$$

para cualesquiera números reales x, y.

Soluciones

1. Determinar todos los números de cuatro cifras $n = \overline{abcd}$ tales que al insertar un dígito 0 en cualquier posición se obtiene un múltiplo de 7.

Solución. Comenzamos observando que el número que resulta de insertar un 0 al final de n es $10n$, que al ser múltiplo de 7 obliga a que n también lo sea. De hecho, son múltiplos de 7 los siguientes cinco números:

$$\begin{aligned}
n = \overline{abcd} &= 1000a + 100b + 10c + d \\
x = \overline{a0bcd} &= 10000a + 100b + 10c + d \\
y = \overline{ab0cd} &= 10000a + 1000b + 10c + d \\
z = \overline{abc0d} &= 10000a + 1000b + 100c + d \\
w = \overline{abcd0} &= 10000a + 1000b + 100c + 10d
\end{aligned}$$

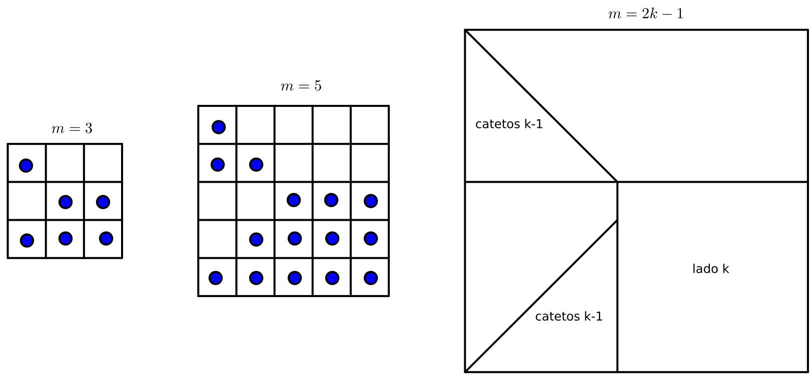
Como n, x son múltiplos de 7, también lo es su diferencia $x - n = 9000a$, y puesto que 9000 no es múltiplo de 7, debe serlo a. Al ser n un número de 4 cifras, se tiene que $a \neq 0$ y necesariamente debe ser $\boxed{a = 7}$.
De forma similar, $y - n = 9000a + 900b$ es múltiplo de 7, y sabemos que a es múltiplo de 7, luego $900b$ es múltiplo de 7. Y como 900 no es múltiplo de 7, debe serlo b, y se deduce que $\boxed{b = 0 \text{ o } b = 7}$.
Análogamente, razonando con $z - n = 9000a + 900b + 90c$ se obtiene que $\boxed{c = 0 \text{ o } c = 7}$, e insertando las condiciones de ser a, b, c múltiplos de 7 en el número de partida n deducimos que también $\boxed{d \text{ es } 0 \text{ o } 7}$.
Así pues, los números buscados son todos los que empiezan por 7 y las restantes cifras son 0 o 7:

$$7000, 7007, 7070, 7077, 7700, 7707, 7770, 7777.$$

2. Determinar todas las parejas de enteros positivos (m, n) para los cuales es posible colocar algunas piedras en las casillas de un tablero de m filas y n columnas, no más de una piedra por casilla, de manera que todas las columnas tengan la misma cantidad de piedras, y no existan dos filas con la misma cantidad de piedras.

Solución 1. Veremos que las soluciones son todas las parejas (m, n) con $n \geq m$ si m es impar, o $n \geq m - 1$ si m es par.
Para $m = 1$, es inmediato que cualquier tablero $(1, n)$ es posible, por ejemplo llenándolo completamente de piedras.
Nótese la condición necesaria $n \geq m - 1$: para m filas son necesarias al menos $m - 1$ columnas, ya que $m - 1$ es la menor cantidad de piedras que puede contener la fila que tenga más piedras.
Sea $m \geq 3$ impar. Veamos que no es posible alcanzar el caso límite $n = m - 1$. En efecto, para llenar un tablero $(m, m - 1)$ con distintas cantidades de piedras en cada fila, es necesario que las filas tengan $0, 1, \ldots, m - 1$ piedras, en algún orden. El número

total de piedras es $\frac{(m-1)m}{2}$, y debe ser igual a $t(m-1)$, siendo t la cantidad de piedras de cada columna. Como la igualdad $\frac{m}{2} = t$ es imposible cuando m es impar, en este caso el número de columnas debe ser $n \geq m$.
Una solución válida para el tablero (m, m), para $m = 2k - 1$, se obtiene siguiendo el esquema de la siguiente figura:

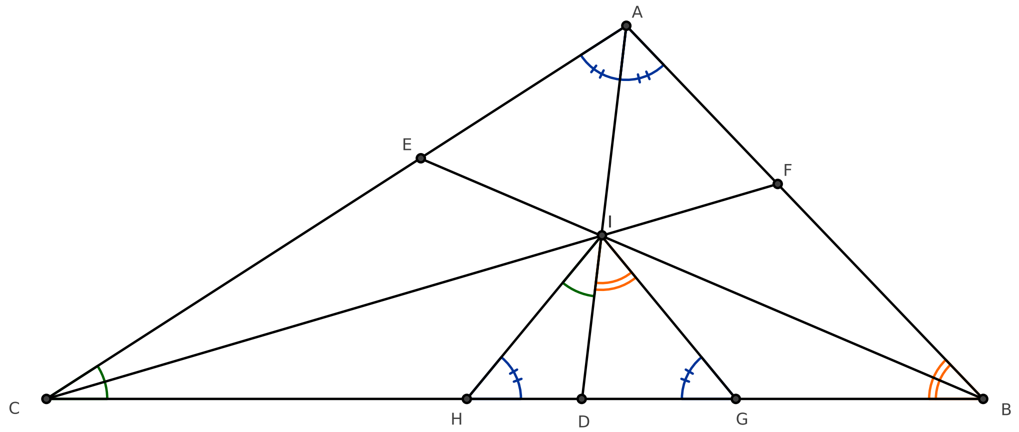


Las piedras se colocan en dos triángulos rectángulos isósceles de catetos $k - 1$, y en un cuadrado de lado k. Las cantidades de piedras en las filas son $1, 2, \ldots, k-1, k, k+1, \ldots, 2k-1$, y cada columna tiene k piedras.
Para $m \geq 2$ par, el tablero $(m, m-1)$ se resuelve partiendo de una solución del tablero $(m-1, m-1)$ y añadiendo una fila vacía.
Finalmente, toda solución para un tablero (m, n) puede extenderse a un tablero $(m, n+1)$, de esta manera: si todas las columnas tienen t piedras, colocamos t piedras en la nueva columna, en las posiciones correspondientes a las t filas que más piedras tenían. Así, las columnas siguen teniendo t piedras cada una, y sigue sin haber dos filas con el mismo número de piedras. Repitiendo las veces que haga falta la operación de paso de (m, n) a $(m, n+1)$, se resuelven todos los tableros (m, n) con $n \geq m$ si m es impar, o $n \geq m - 1$ si m es par.

Solución 2. Similar a la anterior, utilizando el siguiente procedimiento para rellenar las casillas en el tablero (m, m), cuando m es impar.
Vamos a colocar k piedras en la fila $k \in [1, n]$. Empezamos por la esquina superior izquierda colocando una piedra. Si la fila tiene todas las piedras que vamos a colocar, seguimos por la casilla inmediatamente abajo a la derecha (si estamos a la derecha del todo, seguimos por la izquierda del todo). Si la fila aún no tiene todas las piedras, entonces seguimos colocando piedras inmediatamente a la derecha de la anterior (la misma convención aplica en el borde). Evidentemente cuando hayamos terminado la última fila, todas las filas tendrán el número deseado de piedras. Además, como el número total de piedras es divisible por el número de columnas, habrá el mismo número de piedras en cada columna.

3. En el triángulo ABC con lado mayor BC, las bisectrices se cortan en I. Las rectas AI, BI, CI cortan a BC, CA, AB en los puntos D, E, F, respectivamente. Se consideran puntos G y H en los segmentos BD y CD, respectivamente, tales que $\angle GID = \angle ABC$ y $\angle HID = \angle ACB$. Probar que $\angle BHE = \angle CGF$.

Solución. Comenzamos con una figura, donde se señalan algunas igualdades de ángulos que inmediatamente justificaremos.



Nótese que los triángulos ABD y GID tienen un ángulo común en D y ángulos iguales en B y en I, por lo que sus terceros ángulos deben coincidir, es decir $\angle DGI = \angle DAB$. De forma análoga, razonando con los triángulos ACD y HID se obtiene $\angle DHI = \angle DAC$.
Los triángulos BIA y BIH resultan ser congruentes por tener dos ángulos iguales y un lado común, esto revela que los puntos A y H son simétricos con respecto a la recta BI, y de forma similar A y G son simétricos respecto de CI.
Las simetrías que acabamos de establecer prueban que:

$$\angle BHE = BAE \quad \text{y} \quad \angle CGF = \angle CAF,$$

y queda demostrada la igualdad de ángulos que se pedía.
Nota: otras soluciones pueden probar y utilizar que los cuadriláteros $ABGI$ y $ACHI$ son inscriptibles (cíclicos), y/o que G, H son puntos de la circunferencia de centro I que pasa por A.

4. Al desarrollar $(1 + x + x^2)^n$ en potencias de x, exactamente tres términos tienen coeficiente impar. ¿Para qué valores de n es esto posible?

Solución. Empezamos estudiando qué efecto tiene sobre los coeficientes de un polinomio multiplicar por $(1 + x + x^2)$:

$$\begin{aligned}(1+x+x^2)^{n+1} &= (1+x+x^2)^n(1+x+x^2)\\ &= (a_o + a_1x + a_2x^2 + a_3x^3 + \cdots)(1+x+x^2)\\ &= a_o + (a_o+a_1)x + (a_o+a_1+a_2)x^2 +\\ &+ (a_1+a_2+a_3)x^3 + \cdots\end{aligned}$$

Si llamamos $\{a_i\}_{i=0}^{2n}$ a los coeficientes de $P_n(x) = (1 + x + x^2)^n$ y $\{b_i\}_{i=0}^{2n+2}$ a los de $P_{n+1}(x)$, entonces para todo i se cumple que $b_i = a_{i-2} + a_{i-1} + a_i$, identificando con 0 los coeficientes que no están definidos. Esto sugiere ordenar los coeficientes de los polinomios $P_n(x)$ en forma de triángulo, empezando con 1 en la cúspide, y donde los restantes coeficientes son la suma de los tres que se sitúan por encima de él. Algo así:

```
                1
             1  1  1
          1  2  3  2  1
       1  3  6  7  6  3  1
    1  4 10 16 19 16 10  4  1
.................................
```

Cada fila es simétrica con extremos 1, y el coeficiente central siempre es impar por ser suma de un impar más dos números iguales.
También vemos que para $n = 1, 2, 4$, $P_n(x)$ tiene 3 coeficientes impares. Veamos que esta situación ocurre para todas las potencias de 2.
Es sencillo comprobar que si $P_n(x)$ tiene 3 coeficientes impares (necesariamente los de grado 0, n y $2n$), lo mismo ocurre para $P_{2n}(x)$:

$$\begin{aligned}(1+x+x^2)^{2n} &= \left((1+x+x^2)^n\right)^2\\ &\equiv (1+x^n+x^{2n})^2 \equiv 1+x^{2n}+x^{4n} \pmod 2,\end{aligned}$$

donde hemos despreciado los "dobles productos", al trabajar módulo 2. Partiendo de que $P_1(x)$ tiene 3 términos impares, el procedimiento anterior de paso de n a $2n$ prueba que $P_n(x)$ tiene 3 términos impares para todo n potencia de 2. Veamos que no existen más valores de n con esta propiedad.
Si n no es potencia de 2, existen una potencia de 2 dada por $a = 2^k$ y un número b, con $0 < b < a$, tales que $n = a + b$. Por lo visto antes, podemos asumir que $P_a(x) \equiv 1 + x^a + x^{2a} \pmod 2$. Entonces:

$$\begin{aligned}P_n(x) = P_b(x)P_a(x) &\equiv P_b(x)(1+x^a+x^{2a}) \pmod 2\\ &\equiv P_b(x) + (\text{términos de grado} \geq a > b)\end{aligned}$$

Además, sabemos que $P_b(x)$ tiene su término central x^b impar, y este término "sobrevive" sin que nadie lo cancele, por lo que $P_n(x)$ tiene al menos 4 términos impares: los de grado 0, b, n y $2n$.

5. En un torneo de ajedrez participan 8 maestros durante 7 días. Cada día se disputan 4 partidas en las cuales participan todos los maestros, y al finalizar el torneo todos se han enfrentado contra todos exactamente una vez. Demostrar que al terminar el quinto día del torneo existe un conjunto de al menos 4 maestros que ya han jugado entre ellos todas las partidas.

Solución. Sea A_1 un maestro cualquiera. Llamamos B_1 a su rival del día 6, A_2 al rival de B_1 el día 7, B_2 al rival de A_2 el día 6, y así sucesivamente, hasta que se cierre un ciclo:

$$A_1 \xrightarrow{6} B_1 \xrightarrow{7} A_2 \xrightarrow{6} B_2 \xrightarrow{7} \cdots \xrightarrow{7} A_n \xrightarrow{6} B_n \xrightarrow{7} A_1,$$

donde hemos indicado con 6 y 7 el día en que se juega la partida. Nótese que el adversario de A_1 el día 7 no puede ser uno de los A_n, con $n \geq 2$, pues A_n ya ha jugado contra B_{n-1} el mismo día 7. La longitud del ciclo formado es entonces par, y pueden darse estas situaciones:

$$A_1 \xrightarrow{6} B_1 \xrightarrow{7} A_2 \xrightarrow{6} B_2 \xrightarrow{7} A_3 \xrightarrow{6} B_3 \xrightarrow{7} A_4 \xrightarrow{7} B_4 \xrightarrow{7} A_1 \qquad (1)$$

$$A_1 \xrightarrow{6} B_1 \xrightarrow{7} A_2 \xrightarrow{6} B_2 \xrightarrow{7} A_3 \xrightarrow{6} B_3 \xrightarrow{7} A_1 \qquad (2)$$

$$A_1 \xrightarrow{6} B_1 \xrightarrow{7} A_2 \xrightarrow{6} B_2 \xrightarrow{7} A_1 \qquad (3)$$

En la situación (1), resulta que durante los días 6 y 7 no ha habido partidas entre los elementos del conjunto $\{A_1, A_2, A_3, A_4\}$, por lo tanto han debido completar todos sus enfrentamientos durante los primeros 5 días, y lo mismo ocurre para el conjunto $\{B_1, B_2, B_3, B_4\}$.
La situación (2) no puede darse porque deja una sola forma (no permitida) de completar las jornadas 6 y 7: ambas con el enfrentamiento entre los dos maestros que aún no han intervenido.

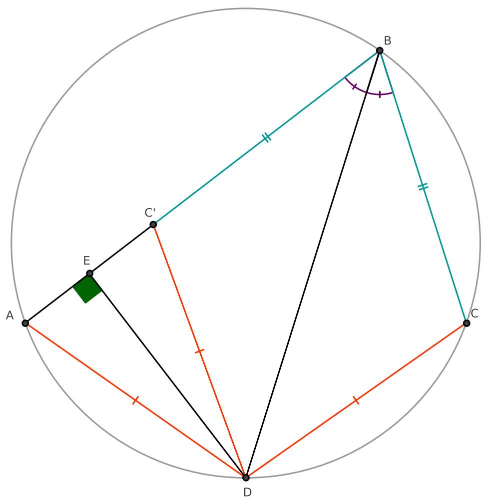
Finalmente, (3) sí puede ocurrir, y las jornadas 6 y 7 se completarían de esta manera, etiquetando adecuadamente los maestros:

$$A_3 \xrightarrow{6} B_3 \xrightarrow{7} A_4 \xrightarrow{6} B_4 \xrightarrow{7} A_3.$$

Los conjuntos $\{A_1, A_2, A_3, A_4\}$, $\{A_1, A_2, B_3, B_4\}$, $\{B_1, B_2, A_3, A_4\}$ y $\{B_1, B_2, B_3, B_4\}$ cumplen la condición pedida en el enunciado: al no haber enfrentamientos internos durante las jornadas 6 y 7, deben haberse completado las partidas entre los días 1 y 5.

6. $ABCD$ es un cuadrilatero convexo verificando $AB > BC$, $CD = DA$ y $\angle ABD = \angle DBC$. Sea E el punto de la recta AB tal que $\angle DEB = 90^circ$. Probar que $AE = \dfrac{AB - BC}{2}$.

Solución 1. (sacándose de la manga un punto auxiliar). Sea C' el punto simétrico de C con respecto a la recta BD, lo incluimos en la figura.



Por simetría se tiene que $\angle C'BD = \angle CBD$, $BC' = BC$ y $DC' = DC$. En particular C' está en el segmento AB y cumple que $AC' = AB - BC$.
Además, el triángulo DAC' es isósceles con $DA = DC'$, por lo tanto la altura y la mediana de D deben coincidir, es decir:

$$AE = \frac{AC'}{2} = \frac{AB - BC}{2}.$$

Solución 2. Aplicando el teorema del coseno en los triángulos ABD y BCD,

$$AD^2 = AB^2 + BD^2 - 2AB \cdot BD \cos(\alpha),$$
$$CD^2 = BC^2 + BD^2 - 2BC \cdot BD \cos(\alpha),$$

donde hemos denotado $\alpha = \angle ABD = \angle DBC$. Como $AD = CD$ y $AB \neq BC$, se tiene igualando ambas expresiones que

$$(AB + BC)(AB - BC) = AB^2 - BC^2 = 2BD(AB - BC)\cos(\alpha),$$
$$BE = BD\cos(\alpha) = \frac{AB + BC}{2}.$$

Luego $AE = AB - BE = \frac{AB-BC}{2}$.

7. Demostrar que todos los números racionales pueden expresarse como suma de algunas fracciones de la forma $\dfrac{n-1}{n+2}$, con $n \geq 0$ entero, admitiendo repetir sumandos.

Solución 1. Llamamos $f(n) = \frac{n-1}{n+2}$, y denotamos por A al conjunto de números que pueden expresarse como suma de elementos $f(n)$, para ciertos $n \geq 0$. Probaremos que $A = \mathbb{Q}$.
Todo número racional puede escribirse como un cociente $\frac{k}{n}$, con k entero (positivo, negativo o nulo) y n estrictamente positivo. Ahora bien, A tiene la propiedad de que si x, y están en A, la suma $x + y$ también lo está. Por lo tanto, para ver que $A = \mathbb{Q}$ será suficiente probar que A contiene a los números $\dfrac{1}{n}$ y $\dfrac{-1}{n}$, para todo $n \geq 1$.
Demostraremos que 1 y -1 están en A, lo cual luego necesitaremos aplicar en momentos oportunos:
•$f(0) = \frac{-1}{2}$, por lo que $f(0) + f(0) = -1 \in A$.
•$f(2) = \frac{1}{4}$, luego $4f(2) = 1 \in A$.
Ahora sí vamos a la caza de las fracciones $\dfrac{1}{n}$ y $\dfrac{-1}{n}$ para $n \geq 2$:

$$f(3n-2) = \frac{3n-3}{3n} = 1 - \frac{1}{n} \in A \Rightarrow f(3n-2) + 2f(0) = \frac{-1}{n} \in A.$$

En consecuencia, sumando $n-1$ veces $\frac{-1}{n} \in A$ llegamos a que $\frac{1-n}{n} \in A$, y sumando $1 \in A$ obtenemos que $\frac{1}{n} \in A$, lo que nos faltaba para terminar el problema.

Solución 2. Llamemos $q_n = \frac{n-1}{n+2}$. Se tiene entonces que $q_1 = 0$, $q_0 = -\frac{1}{2}$ y $q_4 = \frac{1}{2}$. En particular, es posible obtener de la forma deseada el 0 y todos los racionales de la forma $\frac{u}{2}$ (incluyendo que u sea negativo, o que u sea par con lo que el racional será entero), tomando la suma de $|u|$ veces q_0 si $u < 0$ y u veces q_4 si $u > 0$. Podemos ahora completar el problema escribiendo cada racional no nulo en la forma $\frac{u}{v}$ con v entero positivo y u entero no nulo, y por inducción sobre v. Ya hemos visto que para $v = 1, 2$ se puede obtener el racional como suma de q_i's, luego los basta con probar que si para $v = 1, 2, \ldots, m-1$ se puede obtener cada racional como suma de q_i's, también se puede para $v = m \geq 3$.
Si $m \geq 3$ es coprimo con 3, tomemos $n = m-2$, con lo que $q_n = \frac{m-3}{m}$, y como m es coprimo con 3 y $m - (m-3) = 3$, se tiene que $m-3$ y m son coprimos. Consideremos ahora los siguientes números:

$$\begin{aligned}
\frac{u}{m} &= \frac{u}{m} - 0, \\
\frac{u}{m} - \frac{m-3}{m} &= \frac{u+3}{m} - 1, \\
\frac{u}{m} - 2 \cdot \frac{m-3}{m} &= \frac{u+6}{m} - 2, \\
&\vdots \\
\frac{u}{m} - (m-1) \cdot \frac{m-3}{m} &= \frac{u+3(m-1)}{m} - (m-1).
\end{aligned}$$

Nótese que los m números $u, u+3, u+6, \ldots, u+3(m-1)$ dan todos restos distintos al dividir entre m, pues si dos dieran el mismo resto, se tendría que existen i, j, con $0 \leq i < j \leq m-1$, tales que $3(i-j)$ es múltiplo de m. Pero al ser m coprimo con 3, sería $0 < i - j < m$ múltiplo de m, absurdo. Como hay m tales números, exactamente uno de ellos es múltiplo de m, es decir, existe un $0 \leq k \leq m-1$ tal que $\frac{u}{v} - kq_n$ es una

fracción reducible con denominador m, luego se puede escribir como $\frac{u'}{v'}$ con u' entero y v' entero positivo menor que m. Por hipótesis de inducción esta fracción puede ponerse como suma de q_i's, suma a la que añadiendo k veces q_n se obtiene $\frac{u}{v}$, y hemos terminado en este caso.
Si $m \geq 3$ es múltiplo de 3, entonces es de la forma 3ℓ. Tomemos $n = 9\ell - 2$, con lo que $q_n = \frac{3\ell-1}{3\ell}$, y podemos repetir el razonamiento anterior. Luego también en este caso se puede escribir $\frac{u}{m}$ como suma de q_i's, y hemos terminado el problema por inducción.

8. Determinar todas las funciones f tales que

$$f(xf(y) + y) = f(xy) + f(y)$$

para cualesquiera números reales x, y.

Solución 1. Llamamos (E) a la ecuación del enunciado. Para cada $k \in \mathbb{R}$, la función $f(x) = k$ es solución si y sólo si $k = k + k$, es decir $k = 0$. Así, la función $f(x) = 0$ verifica (E), y es la única solución constante.
Por otra parte, la función identidad $f(x) = x$ también verifica (E), ya que los términos izquierdo y derecho son iguales a $xy + y$, para todo x, y. Probaremos que no hay más soluciones que las ya encontradas $f(x) = 0$ y $f(x) = x$.
Ahora sustituimos $x = 0$ en (E), lo cual resulta en $f(y) = f(0) + f(y)$, y se deduce que $f(0) = 0$.
Si f no fuera la función identidad, existiría al menos un número real a tal que $f(a) \neq a$, con $a \neq 0$, ya que $f(0) = 0$. Para ese valor de a, la ecuación $xf(a) + a = xa$ tiene solución $x = \frac{a}{a-f(a)}$, y sustituyendo en (E) y simplificando $f(xf(a) + a) = f(xa)$ obtenemos que $f(a) = 0$.
A continuación, volvemos a la ecuación (E) con y igual al valor anterior de a, permitiendo que x tome valores arbitrarios:

$$\begin{aligned} f(xf(a) + a) &= f(ax) + f(a) \\ \underbrace{f(0 + a)}_{=0} &= f(ax) + 0. \end{aligned}$$

De aquí se obtiene que $f(ax) = 0$ para todo x. Puesto que $a \neq 0$, el cambio de variable $x = \frac{y}{a}$ revela que $f(y) = 0$ para todo y, caso ya estudiado. Esto confirma que toda solución f debe ser la función identidad o la función nula.

Solución 2. Tomando $x = 0$ se tiene que $f(y) = f(0) + f(y)$, es decir $f(0) = 0$. Claramente, $f(x) = 0$ para todo real x es solución ya que ambos lados de (E) serían nulos. Supondremos en adelante que $f(x) \neq 0$ para al menos un real x.
(1) Probaremos que si $f(r) \neq 0$, entonces $f(r) = r$. En efecto, si fuera $f(r) \neq r$, existe un real $z = \frac{r}{r-f(r)}$ tal que $zf(r) + r = rz$, con lo que tomando $x = z$ e $y = r$ en (E) y llamando $k = zf(r) + r = rz$, se tiene que $f(k) = f(k) + f(r)$, o sea $f(r) = 0$.
(2) Probaremos que si f no es idénticamente nula, entonces es la función identidad. En efecto, sea r tal que $f(r) \neq 0$, en particular $f(r) = r \neq 0$. Para cada real z, veamos que $f(z) = z$. Para ello, sustituimos en (E) los valores $x = \frac{z}{r}$ e $y = r$, con lo que $xf(y) = xy = z$, por tanto

$$f(z + r) = f(z) + r.$$

Claramente, no pueden ser $f(z)$ y $f(z + r)$ nulos a la vez, pues en ese caso se tendría $f(r) = 0$, contradicción. Luego uno de los dos ha de ser no nulo. Pero si $f(z) \neq 0$, por (1) se tiene que $f(z) = z$ y entonces $f(z+r) = z+r$, y si $f(z+r) \neq 0$, entonces $f(z+r) = z+r$ y $f(z) = (z+r) - r = z$. En ambos casos $f(z) = z$, como queríamos. Concluimos entonces que hay dos soluciones: $f(x) = 0$, para la que ambos miembros de (E) valen 0, y $f(x) = x$, para la que ambos miembros de (E) valen $xy + y$.

Solución 3. (1) $f(0) = 0$, lo cual se obtiene sustituyendo $x = 0$ en (E).
(2) Si $y \neq 0$, $f(y) = 0$ implica $f \equiv 0$ (f es idénticamente nula). En efecto, sustituyendo en (E) resulta $f(xy) = 0$ para todo x, por tanto al ser $y \neq 0$ se obtiene que $f \equiv 0$.
(3) A partir de ahora supondremos que f no es idénticamente nula, en particular por (2) se tiene que $f(y) = 0$ implica $y = 0$.
(4) Si $a, b \neq 0$ son tales que $f(a) = -f(b)$, entonces $a = -b$. Para ver esto, sustituimos $x = \frac{b}{a}$, $y = a$:

$$f\left(\frac{b}{a}f(a) + a\right) = f(b) + f(a) = 0,$$

luego en virtud de (3) se tiene $\frac{b}{a}f(a) + a = 0$, es decir $f(a) = -\frac{a^2}{b}$. Intercambiando a y b obtenemos que $f(b) = -\frac{b^2}{a}$, y sumando ambas expresiones:

$$0 = f(a) + f(b) = -\frac{a^2}{b} - \frac{b^2}{a} = -\frac{(a^3 + b^3)}{ab} = -\frac{(a+b)(a^2 - ab + b^2)}{ab},$$

y se deduce que $a = -b$.
(5) Veamos que para todo $y \neq 0$ se tiene $f(y) = y$. Por (3) sabemos que $f(y) \neq 0$, luego podemos definir $x = -\frac{y}{f(y)}$, o sea que $xf(y) + y = 0$. Obtenemos

$$0 = f(0) = f(xf(y) + y) = f(xy) + f(y) = f\left(-\frac{y^2}{f(y)}\right) + f(y),$$

entonces en virtud de (4) tenemos que $\frac{y^2}{f(y)} = y$, es decir $f(y) = y$.
Como en las soluciones anteriores, se concluye que $f \equiv 0$ y $f(x) = x$ son las únicas soluciones de la ecuación funcional.

Problems

1. The vertices, A, B and C, of an equilateral triangle of side 1 are on the surface of a sphere of radius 1 and center O. Let D be the orthogonal projection of A onto the plane, α, determined by B, C and O. Call N one of the intersections with the sphere of the line perpendicular to α through O. Find the measure of the angle $\angle DNO$.
(Note: the orthogonal projection of A onto the plane α is the point of intersection with α of the line passing through A and perpendicular to α.)

2. Given a positive integer n, we define $\lambda(n)$ as the number of positive integer solutions of the equation $x^2 - y^2 = n$. We will say that the number n is *olympic* if $\lambda(n) = 2021$. What is the smallest positive integer that is olympic? And what is the smallest odd positive integer that is olympic?

3. We have 2021 colors and 2021 tokens of each color. We place the 2021 tokens in a row. A token, F, is said to be "bad" if on each side of F there are an odd number of the 2020×2021 tokens that do not share color with F.

(a) Determine the minimum possible number of bad tokens.

(b) If we impose the condition that each token must share color with at least one adjacent token, what is the minimum possible number of bad tokens?

4. Let a, b, c, d be real numbers such that

$$a + b + c + d = 0 \qquad \text{and} \qquad a^2 + b^2 + c^2 + d^2 = 12.$$

Find the minimum and maximum values that the product $abcd$ can take, and determine for which values of a, b, c, d this minimum and maximum are obtained.

5. We have $2n$ bulbs placed in two rows (A and B) and numbered from 1 to n in each row. Some (or none) of the bulbs are on and the rest are off; we say that this is a state. Two states are distinct if there is a bulb that is on in one of them and off in the other. We will say that a state is good if there are the same number of light bulbs on in row A as in row B. Show that the total number of good states, EB, divided by the total number of states, ET, is

$$\frac{EB}{ET} = \frac{3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n \, n!}.$$

6. Let ABC be a triangle with $AB \neq AC$, let I be its incenter, γ its incircle and D be the midpoint of BC. The tangent to γ for D distinct of BC meets γ in E. Prove that AE and DI are parallel.

Solutions

1. The vertices, A, B and C, of an equilateral triangle of side 1 are on the surface of a sphere of radius 1 and center O. Let D be the orthogonal projection of A onto the plane, α, determined by B, C and O. Call N one of the intersections with the sphere of the line perpendicular to α through O. Find the measure of the angle $\angle DNO$.
(Note: the orthogonal projection of A onto the plane α is the point of intersection with α of the line passing through A and perpendicular to α.)

Solution. It is obvious that A, B, C and O are vertices of a regular tetrahedron of edge equal to 1, since the distance between any two of them is 1. Since D is the orthogonal projection of A on the opposite face of the tetrahedron, D is the center of the face BCO. Thus, the distance from D to O (distance from the center of an equilateral triangle of side 1 to one of its vertices) is

$$d(D,O) = \frac{2}{3}\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}.$$

Since the triangle DNO is right-angled at O, the leg OD measures $1/\sqrt{3}$ and the leg ON measures 1, the angle required is

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = 30^\circ.$$

2. Given a positive integer n, we define $\lambda(n)$ as the number of positive integer solutions of the equation $x^2 - y^2 = n$. We will say that the number n is *olympic* if $\lambda(n) = 2021$. What is the smallest positive integer that is olympic? And what is the smallest odd positive integer that is olympic?

Solution. We will distinguish 4 cases, depending on whether n is odd or even and whether n is perfect square or not.

(a) Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be an odd number that is not a perfect square. If $x^2 - y^2 = (x+y)(x-y) = n$, with $x, y > 0$, Then there exist positive integers a, b, with $a > b$ and with the same parity, such that $x + y = a$ and $x - y = b$ (whereby, $x = (a+b)/2$, $y = (a-b)/2$). The ways of writing n as a product of two different numbers of the same parity are, in this case, half the number of divisors. So we will have to look for numbers with 4042 divisors:

$$4042 = (a_1 + 1) \cdots (a_r + 1).$$

The factorization of 4042 as a product of primes is $4042 = 2 \cdot 43 \cdot 47$. Therefore, the choices of natural numbers with 4042 divisors are of the form: $pq^{42}r^{46}$; pq^{2020}; $p^{42}q^{93}$;

$p^{46}q^{85}$; or p^{4041}, where $p.q.r$ are different odd primes. It is immediate to check that the option that gives the lowest number is $3^{46} \cdot 5^{42} \cdot 7$.

(b) Let us now consider the case where n is odd and perfect square. In this case, the number of divisors is odd, and since we exclude the case where the two factoring numbers in the factorization are the same, we have to look for perfect squares with 4043 divisors:

$$4043 = (a_1 + 1) \cdots (a_r + 1).$$

The factorization of 2043 as product of primes is $4043 = 13 \cdot 311$, and we have the options $p^{12}q^{310}$ and p^{4042}. The smallest number is $3^{310} \cdot 5^{12}$, which is larger than the one we found before.

(c) Suppose now that $n = 2^k p_1^{a_1} \cdots p_r^{a_r}$ is even, but not a perfect square. Again, we have to do the decomposition $n = ab$, with a and b of the same parity. This makes a and b have to be even. That is, $k \geq 2$. Therefore, the options for the exponent 2 in divisor are $1, 2, \ldots, k-1$, but never neither 0 nor k. Hence,

$$4042 = (k-1)(a_1 + 1) \cdots (a_r + 1),$$

and the options are: $2^3p^{42}q^{46}$; $2^{44}pq^{46}$; $2^{48}pq42$; $2^{87}p^{46}$; $2^{48}p^{85}$; $2^{95}p^{42}$; $2^{44}q^{93}$; 2^3p^{2020}; $2^{2022}p$; and 2^{4043}. The smallest number is $2^{48} \cdot 3^{42} \cdot 5$, which smaller than the number found in (a).

(d) Finally, we consider the case when n is even and perfect square. That is, $4043 = (k-1)(a_1 + 1) \cdots (a_r + 1)$ and the options are $2^{14}p^{310}$, $2^{312}p^{12}$ y 2^{4044}. The three preceding numbers are larger than the ones found in (c). So, the small olympic number is $2^{48} \cdot 3^{42} \cdot 5$ and the small odd olympic number is $3^{46} \cdot 5^{42} \cdot 7$.

3. We have 2021 colors and 2021 tokens of each color. We place the 2021 tokens in a row. A token, F, is said to be "bad" if on each side of F there are an odd number of the 2020×2021 tokens that do not share color with F.

(a) Determine the minimum possible number of bad tokens.

(b) If we impose the condition that each token must share color with at least one adjacent token, what is the minimum possible number of bad tokens?

Solution. We will say that a token that is not bad is good. We number the colors from 1 to 2021. We say that a token is even (odd) if it occupies an even (odd) position in the row, and we say that it is even-colored (odd-colored) if, among the tokens of its color, it occupies an even (odd) place. If a token is even and even-colored, it means that it has before it in the row an odd number of tokens of its color, and also an odd number of tokens in total. Then it has before it an even number of tokens of other colors, and since 2020×2021 is even, the total number of tokens of other colors, it also has an even number of tokens of other colors behind it. Then an even and even-colored token is good. Similarly, an odd and odd-colored token is good, but all even tokens that are odd-colored are bad and all the tokens that are odd and even-colored are also bad. Note that the number of colored-pairs, even, odd and colored-odd tokens are respectively

$$2021 \cdot 1010 < \frac{2021^2 - 1}{2} < \frac{2021^2 + 1}{2} < 2021 \cdot 1011.$$

Suppose there are exactly u tokens that are bad because they are odd but colored-pairs. This means that there are exactly $\frac{2021^2+1}{2} - u$ tokens that are good because they are odd and odd-colored, then the number of tokens that are bad because they are even and odd-colored is exactly

$$2021 \cdot 1010 - \left(\frac{2021^2+1}{2} - u\right) = 1010 + u.$$

The remaining tokens will be even and colored-pairs and therefore good, and the total number of bad tokens will be equal to $1010 + 2u$.

(a) We can put $u = 0$, and thus have exactly 1010 bad tokens, all of them even and odd-colored, if we first place 2020 tokens of color 1, then 2020 tokens of color 2, and so on up to color 2021, and then the remaining 2021 tokens, one of each color. Note that the first $2020 \cdot 2021$ tokens, all are either even and even-colored, or odd and odd-colored, so good in either case. The last 2021 tokens are each of them color-odd, but of them 1011 are odd and therefore good, and 1010 are even and therefore bad. The minimum, achievable in this way, is then equal to 1010.

(b) Note that each bad token because it is even and odd-colored now has next to it a token of the same color, which will therefore be odd and even-colored, so $u > 0$. Let us assign each even and odd-colored token to the next token of the same color that is odd and even-colored. If an even and odd-colored token is contiguous to two of the same color, we assign it to either of the two. Note that since each odd and even-colored token has at most two neighboring tokens, there are at most $2u$ tokens that can be assigned to u odd and even-colored tokens. But we know that there are exactly $1010 + u$ tokens assigned, then $1010 + u \leq 2u$, and therefore $u \geq 1010$. We conclude that there are at least 3030 bad tokens, reaching this minimum if the assignment is complete. That is, if each bad token, being odd and even-colored, is surrounded by two bad tokens of the same color, and each bad token, being even and odd-colored, is contiguous to only one bad token of the same color. This can be achieved for example by first placing 2018 tokens of color 1, then 2018 tokens of color 2, and so on up to color 2021, then placing the remaining $3 \cdot 2021$ tokens, 3 of each color, in 2021 triplets each consisting of the remaining 3 tokens of each color. The first 2018×2021 tokens are clearly good, and of the 2021 triples at the end of the row, 1011 triples have their first odd and odd-colored token, and thus the 3 triple tokens are good, and 1010 triples have their first even and odd-colored token, and thus the 3 triple tokens are bad. The minimum, achievable in this way, is then equal to 3030.

4. Let a, b, c, d be real numbers such that

$$a + b + c + d = 0 \qquad \text{and} \qquad a^2 + b^2 + c^2 + d^2 = 12.$$

Find the minimum and maximum values that the product $abcd$ can take, and determine for which values of a, b, c, d this minimum and maximum are obtained.

Solution. From the conditions of the statement we have that not all numbers have the same sign. The product $abcd$ will take a positive value when two numbers are positive and two negative, so we will look for the maximum assuming that $a, b > 0$ and $c, d < 0$. Note that

$$2(ab + cd) \leq a^2 + b^2 + c^2 + d^2 = 12, \tag{1}$$

whereby $ab + cd \leq 6$. Using AM-GM inequality, yields

$$(ab) \cdot (cd) \leq \left(\frac{ab + cd}{2}\right)^2 \leq 9. \qquad (2)$$

Equality holds in (2) when $ab = cd = 3$, thus (1) forces $(a - b)^2 = (c - d)^2 = 0$. That is, $a = b$ and $c = d$. Therefore, $a = b = \sqrt{3}$, $c = d = -\sqrt{3}$ and the maximum value of the expression is 9.

To find the minimum value we will assume that $a, b, c > 0$ and that $d < 0$. (If three of the numbers are negative and the other one positive, considering their opposite numbers the value of $abcd$ remains invariant). Therefore, $d = -(a + b + c)$ and

$$a^2 + b^2 + c^2 + d^2 = 2(a^2 + b^2 + c^2 + ab + bc + ca) = 12.$$

This means that

$$(a+b+c)^2 = a^2+b^2+c^2+ab+bc+ca+ab+bc+ca \leq 6+\frac{a^2 + b^2 + c^2 + ab + bc + ca}{2} \leq 9,$$

where we have used the well-known inequality $ab+bc+ca \leq a^2+b^2+c^2$. So, $a+b+c \geq 3$. The problem is equivalent to finding the maximum possible value of $abc(a+b+c)$. Again using the inequality between the arithmetic and geometric means,

$$abc \leq \frac{(a + b + c)^3}{27} \qquad \text{therefore} \qquad abc(a + b + c) \leq \frac{(a + b + c)^4}{27} \leq 3.$$

Thus, the minimum value is -3 and it is attained in the cases $(3, -1, -1, -1)$ and $(1, 1, 1, -3)$ and their permutations.

5. We have $2n$ bulbs placed in two rows (A and B) and numbered from 1 to n in each row. Some (or none) of the bulbs are on and the rest are off; we say that this is a state. Two states are distinct if there is a bulb that is on in one of them and off in the other. We will say that a state is good if there are the same number of light bulbs on in row A as in row B. Show that the total number of good states, EB, divided by the total number of states, ET, is

$$\frac{EB}{ET} = \frac{3 \cdot 5 \cdot 7 \cdots (2n - 1)}{2^n\, n!}.$$

Solution. It is obvious that $ET = 2^{2n}$, since each of the $2n$ bulbs can be either off or on. The number of good states with k bulbs on in each row is $\binom{n}{k}^2$ since there are $\binom{n}{k}$ ways to choose the k lit bulbs in row A and as many ways to choose the k lit bulbs in row B. Consequently,

$$EB = \sum_{k=0}^{n} \binom{n}{k}^2.$$

It is well known that this sum is $EB = \binom{2n}{n}$. In any case, suffice it to observe that a state is good if there are exactly k $(0 \leq k \leq n)$ light bulbs on in row A and exactly $n-k$

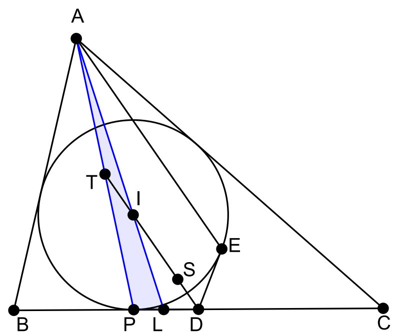
light bulbs off in row B. Each good state is obtained (in only one way) by choosing n bulbs in total and making the chosen ones in row A and the unchosen ones in row B be on. Thus, $EB = \binom{2n}{n}$.

Now, we have

$$\frac{EB}{ET} = \frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n)!}{2^{2n} n!\, n!} = \frac{\left(\prod_{k=1}^{n} 2k\right)\left(\prod_{k=1}^{n}(2k-1)\right)}{2^{2n} n!\, n!}$$
$$= \frac{2^n n!\left(\prod_{k=1}^{n}(2k-1)\right)}{2^{2n} n!\, n!} = \frac{3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n\, n!}.$$

6. Let ABC be a triangle with $AB \neq AC$, let I be its incenter, γ its incircle and D be the midpoint of BC. The tangent to γ for D distinct of BC meets γ in E. Prove that AE and DI are parallel.

Solution. Let P be the tangent point of γ with BC, and let S and T the midpoints of EP and AP, respectively. Clearly ST is parallel to AE. Since DI is the bisector of EP, then S lies on DI. Therefore, the problem reduces to prove that T lies on DI.



Scheme for solving problem 6.

Let L be a point in which the bisector AI cuts BC. Applying Menelau's theorem to triangle APL, we have to prove that

$$\frac{AT}{TP}\frac{PD}{DL}\frac{LI}{IA} = 1.$$

By definition of T, we have $AT/TP = 1$. If a, b and c are the lengths of the sides of $\triangle ABC$, applying the bisector theorem we have $BL/CL = c/b$. Since $BL + LC = a$, then $CL = ab/(b+c)$. Applying the bisector theorem to ALC, we get $\frac{LI}{IA} = \frac{CL}{CA} = \frac{a}{b+c}$. We also have $CD = \frac{a}{2}$ and $CP = \frac{a+b-c}{2}$, whereby

$$\frac{PD}{DL} = \frac{CP - CD}{CL - CD} = \frac{\frac{a+b-c}{2} - \frac{a}{2}}{\frac{ab}{b+c} - \frac{a}{2}} = \frac{b+c}{a}.$$

Substituting, we obtain $\frac{AT}{TP}\frac{PD}{DL}\frac{LI}{IA} = 1$, as we wanted to prove.

Problems

1. The numbers $\{1, 2, \ldots, 80\}$ are written in the blackboard. Gaudí plays the following *solitaire* game. Whenever there are at least two numbers written in the blackboard, he chooses two among them, a and b, erase them, and write instead either the number $8a+b$ or the number $a-6b$. He wins if in the moment when only one number is written in the blackboard, this is 2021. Does he have a winning strategy?

2. Let $ABCD$ be a parallelogram. A variable line drawn through A cuts BC and CD at F and G respectively and cuts the parallel to BD through C at E. Prove that

$$\frac{1}{AF} + \frac{1}{AG} = \frac{2}{AE}.$$

3. Let a, b, n be integers such that $0 < a \leq b < n$. Prove that there exists a prime p that divides both $\binom{n}{a}$ and $\binom{n}{b}$.

4. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2 - bc}{2a^2 + ab + ac} + \frac{b^2 - ca}{2b^2 + bc + ba} + \frac{c^2 - ab}{2c^2 + ca + cb} \leq 0.$$

Solutions

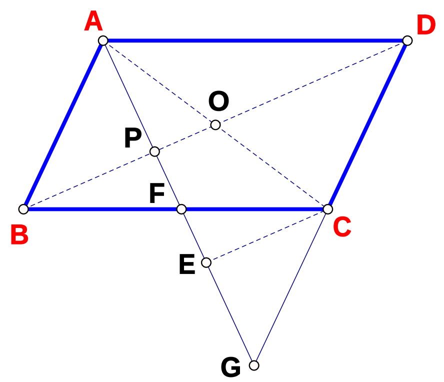
1. *The numbers $\{1, 2, \ldots, 80\}$ are written in the blackboard. Gaudí plays the following solitaire game. Whenever there are at least two numbers written in the blackboard, he chooses two among them, a and b, erase them, and write instead either the number $8a+b$ or the number $a-6b$. He wins if in the moment when only one number is written in the blackboard, this is 2021. Does he have a winning strategy?*

Solution. Consider the numbers modulo 7. Then, in each step, two numbers a and b are chosen and replaced either by $8a + b \equiv a + b$ modulo 7 or by $a - 6b \equiv a + b$ modulo 7. That is, working modulo 7, the operation consists on replacing two numbers a and b by $a + b$. Then, the resulting number modulo 7 once the game is finished will be $1 + 2 + \ldots + 80 \equiv 6$ modulo 7, and 2021 is congruent with 5. Therefore, he cannot win the game.

2. *Let $ABCD$ be a parallelogram. A variable line drawn through A cuts BC and CD at F and G respectively and cuts the parallel to BD through C at E. Prove that*

$$\frac{1}{AF} + \frac{1}{AG} = \frac{2}{AE}.$$

Solution. Let $\{O\} = AC \cap BD$ and $\{P\} = AE \cap BD$. Since OP is the mid parallel to $\triangle ACE$ then $AE = 2AP$. We have the following triangle similarities: $\triangle APD \sim \triangle FPB$, $\triangle ABF \sim \triangle GCF$, $\triangle GCF \sim \triangle GCA$, from which we obtain



Scheme for solving problem 2

$$\frac{AP}{AD} = \frac{FP}{FB} \Leftrightarrow \frac{1}{AP} = \frac{FB}{FP \cdot AD} = \frac{FB}{FP \cdot BC},$$

$$\frac{AF}{BF} = \frac{GF}{CF} \Leftrightarrow \frac{1}{AF} = \frac{CF}{BF \cdot GF},$$

$$\frac{AG}{AD} = \frac{GF}{CF} \Leftrightarrow \frac{1}{AG} = \frac{CF}{AD \cdot GF} = \frac{CF}{BC \cdot GF}.$$

Then,

$$\frac{1}{AF} + \frac{1}{AG} = \frac{CF}{FG}\left(\frac{1}{BF} + \frac{1}{BC}\right) = \frac{CF}{FG} \cdot \frac{BF + BC}{BF \cdot BC} \cdot \frac{1}{AP} \cdot \frac{PF \cdot BC}{BF}$$

$$= \frac{1}{AP} \cdot \frac{CF}{FG} \cdot \frac{BF + BC}{BF \cdot BC} \cdot \frac{PF}{BF}.$$

Since $\triangle DPA \sim \triangle BPF$ then we have

$$\frac{FB}{PF} = \frac{AD}{AP} = \frac{BC}{AP} \Leftrightarrow FB(AP + PF) = PF(BC + FB)$$

from which

$$\frac{PF}{AP + PF} = \frac{FB}{BC + FB} \Leftrightarrow \frac{BC + FB}{BF} = \frac{AF}{PF}$$

follows. Since $\triangle CFG \sim \triangle BFA$ then we have

$$\frac{CF}{FG} = \frac{BF}{AF} \Leftrightarrow \frac{CF}{FG} \cdot \frac{AF}{BF} = 1.$$

Finally, we have

$$\frac{1}{AF} + \frac{1}{AG} = \frac{1}{AP} \cdot \frac{CF}{FG} \cdot \frac{AF}{PF} \cdot \frac{PF}{BF} = \frac{1}{AP} = \frac{2}{AE},$$

and we are done.

3. *Let a, b, n be integers such that $0 < a \le b < n$. Prove that there exists a prime p that divides both $\binom{n}{a}$ and $\binom{n}{b}$.*

Solution. We argue by contradiction. Suppose that $\binom{n}{a}$ and $\binom{n}{b}$ are relatively prime. We need the following identity:

$$\binom{n}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}$$

valid for $0 \le a \le b \le n$. Indeed, suppose we have a class with n students. Then we may choose $\binom{n}{b}$ committees of b students and for each b students we have $\binom{n}{a}$ subcommittees with a girls. Thus, the total number of students may be counted (committed) in two ways:

$$\binom{n}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}.$$

From the preceding, we get that

$$\binom{n}{a} \Bigm| \binom{n}{b}\binom{b}{a}.$$

But since $\binom{n}{a}$ and $\binom{n}{b}$ are coprime, it follows that $\binom{n}{a}$ divides $\binom{b}{a}$ which is impossible because it is clear that $\binom{b}{a} < \binom{n}{a}$. Thus,

$$GCD\left(\binom{n}{a}, \binom{n}{b}\right) > 1,$$

and we are done.

4. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2-bc}{2a^2+ab+ac}+\frac{b^2-ca}{2b^2+bc+ba}+\frac{c^2-ab}{2c^2+ca+cb}\le 0.$$

Solution. First, we write the inequality claimed as

$$\sum_{cyclic}\frac{a}{2a+b+c}-\sum_{cyclic}\frac{bc}{a(2a+b+c)}\le 0.$$

Applying Cauchy's inequality to the vectors $\vec{u}=\left(\sqrt{a+b},\sqrt{a+c}\right)$ and $\vec{v}=\left(\frac{1}{\sqrt{a+b}},\frac{1}{\sqrt{a+c}}\right)$, yields $4\le\left(\frac{1}{a+b}+\frac{1}{a+c}\right)(2a+b+c)$, and we obtain

$$\frac{a}{2a+b+c}\le\frac{1}{4}\left(\frac{a}{a+b}+\frac{a}{a+c}\right).$$

Likewise, we get

$$\frac{b}{a+2b+c}\le\frac{1}{4}\left(\frac{b}{b+c}+\frac{b}{b+a}\right)\quad\text{and}\quad\frac{c}{a+b+2c}\le\frac{1}{4}\left(\frac{c}{a+c}+\frac{c}{b+c}\right).$$

Adding up the preceding inequalities yields,

$$\sum_{cyclic}\frac{a}{2a+b+c}\le\frac{1}{4}(1+1+1)=\frac{3}{4}.$$

Applying again Cauchy's inequality to the vectors

$$\vec{u}=\left(\sqrt{\frac{bc}{a(2a+b+c)}},\sqrt{\frac{ca}{b(a+2b+c)}},\sqrt{\frac{ab}{c(a+b+2c)}}\right)$$

and

$$\vec{v}=\left(\sqrt{bca(2a+b+c)},\sqrt{cab(a+2b+c)},\sqrt{abc(a+b+2c)}\right),$$

yields

$$(bc+ca+ab)^2\le\left(\frac{bc}{a(2a+b+c)}+\frac{ca}{b(a+2b+c)}+\frac{ab}{c(a+b+2c)}\right)4abc(a+b+c).$$

Then,

$$\sum_{cyclic}\frac{bc}{a(2a+b+c)}\ge\frac{(ab+bc+ca)^2}{4abc(a+b+c)}\ge\frac{3}{4}.$$

Indeed,

$$\frac{(ab+bc+ca)^2}{4abc(a+b+c)}\ge\frac{3}{4}\Leftrightarrow a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)\ge 0$$

which immediately follows from the well-known inequality $x^2+y^2+z^2\ge xy+yz+zx$. Finally, we have

$$\sum_{cyclic}\frac{a}{2a+b+c}-\sum_{cyclic}\frac{bc}{a(2a+b+c)}\le\frac{3}{4}-\frac{3}{4}=0.$$

Equality holds when $a=b=c$, and we are done.

Problems

1. According to Anna, the number 2021 is fantabulous. She states that if any element of the set $\{m, 2m+1, 3m\}$ is fantabulous for a positive integer m, then they are all fantabulous. Is the number $(2021)^{2021}$ fantabulous?

2. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that the equation

$$f(xf(x) + y) = f(y) + x^2$$

holds for all rational numbers x and y.

3. Let ABC be a triangle with an obtuse angle at A. Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.

4. Let ABC be a triangle with incentre I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A in the line EF lies on the line BC.

5. A plane has a special point O called the origin. Let P be a set of 2021 points in the plane, such that

- no three points in P lie on a line and
- no two points in P lie on a line through the origin.

A triangle with vertices in P is *fat*, if O is strictly inside the triangle. Find the maximum number of fat triangles.

8. Does there exist a nonnegative integer a for which the equation

$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \ldots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$

has more than one million different solutions (m, n) where m and n are positive integers?

Problems

1. Determine the smallest positive integer M with the following property: For every choice of integers a, b, c, there exists a polynomial $P(x)$ with integer coefficients so that

$$P(1) = aM \qquad \text{and} \qquad P(2) = bM \qquad \text{and} \qquad P(4) = cM.$$

2. For every sequence $p_1 < p_2 < \ldots < p_8$ of eight prime numbers, determine the largest integer N for which the following equation has no solution in positive integers $x_1, x_2, \ldots, x_8$:

$$p_1 p_2 \ldots p_8 \left(\frac{x_1}{p_1} + \frac{x_2}{p_2} + \ldots + \frac{x_8}{p_8} \right) = N.$$

3. Equilateral triangle ABC is inscribed in circle w. Points F and E are chosen on sides AB and AC, respectively, so that $AngleABE + \angle LAF = 60°$. Circumscribed circle of MFE intersects circle w at point D. Rays DE and DF intersect line BC at points X and Y, respectively. Prove that the center point of inscribed circle of $\triangle DXY$ does not depend on the choice of points F and E.

4. Let x_1, x_2, x_3, x_4, x_5 be nonnegative real numbers. If $x_1 \leq 4$, $x_1 + x_2 \leq 13$, $x_1 + x_2 + x_3 \leq 29$, $x_1 + x_2 + x_3 + x_4 \leq 54$ and $x_1 + x_2 + x_3 + x_4 + x_5 \leq 90$, then prove that $\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} + \sqrt{x_5} \leq 20$.

Problems

1. Consider a permutation $(a_1, a_2, \ldots, a_{2021})$ of the numbers $(1, 2, \ldots, 2021)$. Find the minimum and the maximum value that can take the expression

$$a_1^2 + \ldots + a_8^2 + a_9 a_{10} \ldots a_{2021}.$$

2. Let ABC be an acute-angled triangle and let H be its orthocenter. If h_a, h_b and h_c are the lengths of the corresponding altitudes, then prove that

$$\frac{AH + BH + CH}{2} \leq \max\{h_a, h_b, h_c\}.$$

3. Let a, b and c be positive integers for which $a \mid b^2, b \mid c^2,\ c \mid a^2$ hold. Determine whether or not the following statements are true or false, justifying your answer:

(a) All numbers a, b, c that satisfy the above conditions also verify that abc divides $(a + b + c)^6$?

(b) All numbers a, b, c that satisfy the above conditions verify that abc divides $(a + b + c)^7$?

4. Let x, y be two relatively prime positive integers, and $p \geq 3$ be a prime number. Compute

$$\gcd\left(x + y, \frac{x^p + y^p}{x + y}\right).$$

Solutions

1. Consider a permutation $(a_1, a_2, \ldots, a_{2021})$ of the numbers $(1, 2, \ldots, 2021)$. Find the minimum and the maximum value that can take the expression

$$a_1^2 + \ldots + a_8^2 + a_9 a_{10} \ldots a_{2021}.$$

Solution. Consider the sequence for which that value is maximum. Without loss of generality we may assume that $a_1 > a_2 > \ldots > a_8$ and that $a_9 < \ldots < a_{2021}$. Let us suppose that $a_1 > a_9$ and reach a contradiction by swapping the values of a_1 and a_9. In particular, we will prove that

$$a_9^2 + \ldots + a_8^2 + a_1 a_{10} \cdots a_{2021} > a_1^2 + \ldots a_8^2 + a_9 a_{10} \cdots a_{2021}.$$

This is equivalent to

$$(a_1 - a_9) a_{10} \cdots a_{2021} > (a_1 + a_9)(a_1 - a_9).$$

But once we divide by $a_1 - a_9$, this follows by observing that

$$a_{10} \cdots a_{2021} > 2011! > 4042 > a_1 + a_9.$$

Then, the maximum value is

$$1^2 + 2^2 + \ldots + 8^2 + 9 \cdot 10 \cdots 2021.$$

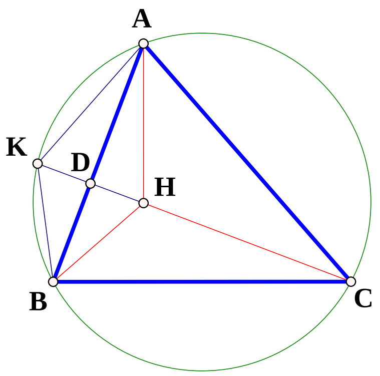
The same argument shows that the minimum is attained when the numbers $(a_1, \ldots, a_8)$ correspond to $(2014, \ldots, 2021)$. Then, the minimum values is

$$2014^2 + \ldots + 2021^2 + 1 \cdot 2 \cdots 2013.$$

2. Let ABC be an acute-angled triangle and let H be its orthocenter If h_a, h_b and h_c are the lengths of the corresponding altitudes, then prove that

$$\frac{AH + BH + CH}{2} \leq \max\{h_a, h_b, h_c\}.$$

Solution 1. Let $\angle C$ be the smallest angle, so that $CA \geq AB$ and $CB \geq AB$. In this case h_c the altitude through C is the longest one. Let the altitude through C meet AB in D and let H be the orthocentre of $\triangle ABC$. Let CD extended meet the circumcircle of $\triangle ABC$ in point K.



Scheme for solving problem 2.

We have $h_c = CD$ so that the inequality to be proven is

$$AH + BH + CH \leq 2CD.$$

On account that $CD = CH + HD$, the above reduces to $AH + BH \leq CD + HD$. Since K is the symmetric of H respect to D then we have $HD = DK$ and also that right triangles DBK and DBH are similar. So, $BH = BK$. Likewise, we have $AH = AK$.

Thus we need to prove that $AK + BK \leq CK$. Applying Ptolemy's theorem to the cyclic quadrilateral $BCAK$, we get

$$AB \cdot CK = AC \cdot BK + BC \cdot KA.$$

On account that $CA \geq AB$ and $CB \geq AB$, we have

$$AB \cdot CK \geq AB \cdot BK + AB \cdot AK$$

from which $AK + BK \leq CK$ follows.

Solution 2. We first note that $AH = 2R\cos\alpha$, and similarly for the other lengths BH and CH. Using that $h_a = c\sin\beta = b\sin\gamma$, the statement is equivalent to

$$R(\cos\alpha + \cos\beta + \cos\gamma) \leq 2R\max\{\sin\beta\sin\gamma, \sin\gamma\sin\alpha, \sin\alpha\sin\beta\}.$$

Without loss of generality, we may assume that $a \geq b \geq c$, and therefore $\sin\alpha \geq \sin\beta \geq \sin\gamma$. By virtue of the cosine rule, the inequality we want to prove may now be rewritten as

$$\frac{a^2b + a^2c + b^2c + b^2a + c^2a + c^2b - a^3 - b^3 - c^3}{2abc} \leq 2\sin\alpha\sin\beta = \frac{8S^2}{abc^2},$$

where the last equality follows from the relations $a = 2R\sin\alpha$, $b = 2R\sin\beta$ and $abc = 4SR$. Here, as usual, S is the area of the triangle. Using now Heron's formula, we may write $16S^2$ as a function of the sides a, b and c, and the inequality is then equivalent to

$$a^2bc + a^2c^2 + b^2c^2 + b^2ca + c^3a + c^3b - a^3c - b^3c - c^4 \leq 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4.$$

But rearranging the terms, this is the same as

$$a^3(c - a) + b^3(c - b) + bc^2(b - c) + ac^2(a - c) + ab^2(a - c) + ba^2(b - c)$$

$$= (a-c)a(b^2+c^2-a^2) + (b-c)b(b^2+c^2-a^2) \geq 0,$$

which clearly holds since $a, b \geq c$ (by assumption), and $b^2+c^2 \geq a^2$ since the triangle is acute-angled.

Solution 3. Let D be the foot of the altitude from A and let a, b and c be the lengths of the sides BC, CA and AB, respectively. The orthocenter H is on the segment AD because the triangle ABC is acute, so

$$a \cdot AH = a \cdot (AD - HD) = a \cdot h_a - a \cdot HD = 2[ABC] - 2[HBC],$$

where we used that base times height is double the area.

Similarly, $b \cdot BH = 2[ABC] - 2[AHC]$ and $c \cdot CH = 2[ABC] - 2[ABH]$. Adding up the three equalities and using that $[HBC] + [AHC] + [ABH] = [ABC]$ because the first three triangles cover the last one, we obtain

$$a \cdot AH + b \cdot BH + c \cdot CH = 6[ABC] - 2([HBC] + [AHC] + [ABH]) = 4[ABC]$$

Suppose, without loss of generality, that a is smaller than (or equal to) b and c. Then, we have that

$$a(AH + BH + CH) \leq a \cdot AH + b \cdot BH + c \cdot CH = 4[ABC],$$

so

$$\frac{AH + BH + CH}{2} \leq \frac{2[ABC]}{a} = h_a = \max\{h_a, h_b, h_c\}$$

The last equality follows from BC, of length a, being the shortest side, meaning that $h_a = \frac{2[ABC]}{a}$ is the longest height. Since BH and CH are non-zero, the inequality of the statement holds with equality only when a is equal to b and c, which means that ABC is equilateral.

As a bonus, with a similar argument one can also prove the other bound:

$$\min\{h_a, h_b, h_c\} \leq \frac{AH + BH + CH}{2} \leq \max\{h_a, h_b, h_c\}$$

3. Let a, b and c be positive integers for which $a \mid b^2, b \mid c^2, \ c \mid a^2$ hold. Determine whether or not the following statements are true or false, justifying your answer:

(a) All numbers a, b, c that satisfy the above conditions also verify that abc divides $(a+b+c)^6$?

(b) All numbers a, b, c that satisfy the above conditions verify that abc divides $(a+b+c)^7$?

Solution 1. (a) The answer is NOT. Indeed, triple $a = 4, b = 2, c = 16$ is a counterexample. We can see that $4 \mid 2^2, 2 \mid 16^2$, and $16 \mid 4^2$, but $abc = 2^7$ does not divide $(a+b+c)^6 = 22^6$.

(b) Expanding $(a+b+c)^7$, we get the following sum:

$$(a+b+c)^7 = \sum_{i+j+k=7} \binom{7}{i, j, k} a^i b^j c^k = \sum_{i+j+k=7} \frac{7!}{i!j!k!} a^i b^j c^k,$$

where $0 \le i, j, k \le 7$ and each $\binom{7}{i,j,k}$ is some positive integer. We will show that each term in the above sum is divisible by abc. We distinguish the following cases:

1. If $i, j, k \ge 1$, then $abc \mid a^i b^j c^k$ obviously holds.

2. If two of i, j, k are 0, then we can assume $i = j = 0$ and so $k = 7$. From $b \mid c^2$ we get $b^2 \mid c^4$ and so $a \mid b^2 \mid c^4$. Therefore $abc \mid c^4 \cdot c^2 \cdot c = c^7 = a^i b^j c^k$. The other two cases follow by an analogue reasoning.

3. If exactly one of i, j, k is 0, then we can assume it is i. In this case $j + k = 7$. If $j \ge 3$, then $ab \mid b^j$ and $c \mid c^k$, therefore $abc \mid b^j c^k = a^i b^j c^k$. If $j \le 2$, then $k \ge 5$. In this case $ac \mid c^4 \cdot c \mid c^k$ and $b \mid b^j$, so again $abc \mid b^j c^k = a^i b^j c^k$.

Solution 2. We start by proving statement **(b)** first, and we will show that the statement is true.

From the conditions $a \mid b^2$, $b \mid c^2$, $c \mid a^2$, it is immediate to see that if p is a prime number that divides one of a, b, c, then it will divide all of them.

So, we can write these numbers as $a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $b = p_1^{\beta_1} \cdots p_k^{\beta_k}$, $c = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$, for some different prime numbers $p_1, \dots, p_k$, where each of the exponents will be a strictly positive integer.

Let $g = \gcd(a, b, c)$. We can write g as $g = p_1^{m_1} \cdots p_k^{m_k}$, where $m_i = \min(\alpha_i, \beta_i, \gamma_i)$ for each $i \in \{1, \dots k\}$.

Since m_i will be a minimum of three numbers, it will be equal to at least one of them. Assume without loss of generality that $m_i = \gamma_i$. From the conditions of the statement, we will have $\beta_i \le 2\gamma_i = 2m_i$ since $b \mid c^2$, and $\alpha_i \le 2\beta_i \le 4m_i$ since $a \mid b^2$. Therefore, we obtain $\alpha_i + \beta_i + \gamma_i \le 4m_i + 2m_i + m_i = 7m_i$.

Let $A = a/g, B = b/g, C = c/g$. Since $g \mid a$, $g \mid b$, $g \mid c$, these three numbers will be integers. Thus, $(a + b + c)^7 = g^7(A + B + C)^7$, meaning $g^7 \mid (a + b + c)^7$.

Finally, we can see that $abc = p_1^{\alpha_1+\beta_1+\gamma_1} \cdots p_k^{\alpha_k+\beta_k+\gamma_k}$, and so $abc \mid p_1^{7m_1} \cdots p_k^{7m_k}$, meaning that $abc \mid g^7 \mid (a + b + c)^7$, as we wanted to show.

We will see that statement **(a)** is false by giving a counterexample. The previous arguments suggest us to take $b = c^2$, $a = b^2 = c^4$. By taking $c = 2$, meaning that $(a, b, c) = (16, 4, 2)$, we see that $abc = 2^7$, whereas $(a + b + c)^6 = 22^6 = 2^6 \cdot 11^6$, which is the counterexample we were looking for.

4. Let x, y be two relatively prime positive integers and $p \ge 3$ be a prime number. Compute

$$\gcd\left(x + y, \frac{x^p + y^p}{x + y}\right).$$

Solution. We attempt to simplify the problem to the case when $y = 1$. Our goal is to compute

$$\gcd\left(x + 1, \frac{x^p + 1}{x + 1}\right).$$

Factoring gives

$$\frac{x^p+1}{x+1} = x^{p-1} - x^{p-2} + x^{p-3} - x^{p-4} + \ldots - x + 1.$$

In order to calculate

$$\gcd\left(x+1, \frac{x^p+1}{x+1}\right),$$

we attempt to reduce the above expression mod $x+1$. Using the fact that p is an odd prime, we know that $p-1$ is even, therefore

$$\begin{aligned}
\frac{x^p+1}{x+1} &= x^{p-1} - x^{p-2} + \ldots + x^{2a} - x^{2a-1} + \ldots - x + 1 \\
&\equiv (-1)^{p-1} - (-1)^{p-2} + \ldots + (-1)^{2a} - (-1)^{2a-1} + \ldots - x + 1 \\
&\equiv \underbrace{1 + 1 + \ldots + 1 + 1}_{p} \equiv p \pmod{x+1}
\end{aligned}$$

Now, by the Euclidan Algorithm, we have

$$\gcd\left(x+1, \frac{x^p+1}{x+1}\right) = \gcd(x+1, p).$$

Since p is a prime, the above expression can only be equal to 1 or p, depending on x. We have now solved the problem for $y = 1$. We wish to generalize the method to any y.

Using a similar factorization as above, we have

$$\frac{x^p+y^p}{x+y} = x^{p-1} - x^{p-2}y + x^{p-3}y^2 - x^{p-4}y^3 + \ldots - xy^{p-2} + y^{p-1}.$$

In order to invoke the Euclidean Algorithm, we wish to evaluate this expression mod $x+y$. Using the fact that $x \equiv -y \pmod{x+y}$ and that $p-1$ is even, we can simplify as follows:

$$\begin{aligned}
x^{p-1} - x^{p-2}y + \ldots - xy^{p-2} + y^{p-1} &\equiv (-y)^{p-1} - (-y)^{p-2} + \ldots + y^{p-1} \\
&\equiv (-1)^{p-1}\left(\underbrace{y^{p-1} + y^{p-1} + \ldots + y^{p-1}}_{p}\right) \\
&\equiv py^{p-1} \pmod{x+y}
\end{aligned}$$

Therefore, by the Euclidean Algorithm, we arrive at

$$\gcd\left(x+y, \frac{x^p+y^p}{x+y}\right) = \gcd(x+y, py^{p-1}).$$

Now, in the problem statement, it was given that x and y are relatively prime. Hence, similarly, $\gcd(x+y, y) = 1$, and we can simplify the above expression further:

$$\gcd\left(x+y, \frac{x^p+y^p}{x+y}\right) = \gcd(x+y, py^{p-1}) = \gcd(x+y, p) = 1 \text{ or } p,$$

depending on whether p divides $x+y$ or not.

Problems

1. Considere el cuadrilátero convexo $ABCD$. El punto P está en el interior de $ABCD$. Asuma las siguientes igualdades de razones:

$$\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC.$$

Demuestre que las siguientes tres rectas concurren en un punto: la bisectriz interna del ángulo $\angle ADP$, la bisectriz interna del ángulo $\angle PCB$ y la mediatriz del segmento AB.

2. Los números reales a, b, c, d son tales que $a \geq b \geq c \geq d > 0$ y $a + b + c + d = 1$. Demuestre que

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1.$$

3. Hay $4n$ piedritas de pesos $1, 2, 3, \ldots, 4n$. Cada piedrita se colorea de uno de n colores de manera que hay cuatro piedritas de cada color. Demuestre que podemos colocar las piedritas en dos montones de tal forma que las siguientes dos condiciones se satisfacen:

- Los pesos totales de ambos montones son iguales.
- Cada montón contiene dos piedritas de cada color.

4. Sea $n > 1$ un entero. A lo largo de la pendiente de una montaña hay n^2 estaciones, todas a diferentes altitudes. Dos compañías de teleférico, A y B, operan k teleféricos cada una. Cada teleférico realiza el servicio desde una estación a otra de mayor altitud (sin paradas intermedias). Los teleféricos de la compañía A parten de k estaciones diferentes y acaban en k estaciones diferentes; igualmente, si un teleférico parte de una estación más alta que la de otro, también acaba en una estación más alta que la del otro. La compañía B satisface las mismas condiciones. Decimos que dos estaciones están *unidas* por una compañía si uno puede comenzar por la más baja i llegar a la más alta con uno o más teleféricos de esa compañía (no se permite otro tipo de movimientos entre estaciones).
Determinar el menor entero positivo k para el cual se puede garantizar que hay dos estaciones unidas por ambas compañías.

5. Se tiene una baraja de $n > 1$ cartas, con un entero positivo escrito en cada carta. La baraja tiene la propiedad de que la media aritmética de los números escritos en cada par de cartas es también la media geométrica de los números escritos en alguna colección de una o más cartas.
¿Para qué valores de n se tiene que los números escritos en las cartas son todos iguales?

6. Pruebe que existe una constante positiva c para la que se satisface la siguiente afirmación:
Sea $n > 1$ un entero y sea $\mathcal{S}$ un conjunto de n puntos del plano tal que la distancia entre cualesquiera dos puntos diferentes de $\mathcal{S}$ es al menos 1. Entonces existe una recta ℓ separando $\mathcal{S}$ tal que la distancia de cualquier punto de $\mathcal{S}$ a ℓ es al menos $cn^{-1/3}$.
(Una recta ℓ separa un conjunto de puntos $\mathcal{S}$ si ℓ corta a alguno de los segmentos que une dos puntos de $\mathcal{S}$.)

Problems

1. Let $P = \{p_1, p_2, \ldots, p_{10}\}$ be a set of 10 different prime numbers and let A be the set of all integers greater than 1 such that their prime factorizations contain only primes in P. Each element in A is colored in the following way:

(a) each element in P has a distinct color,

(b) if m, $n \in A$, then mn has the same color as either m or n,

(c) for each pair of distinct colors $\mathcal{R}$ and $\mathcal{S}$, there are no j, k, m, $n \in A$ (not necessarily distinct), with j, k colored $\mathcal{R}$ and m, n colored $\mathcal{S}$, such that both j divides m and n divides k.

Show that there is some prime in P such that all of its multiples in A have the same color.

2. Consider an acute triangle ABC, with $AC > AB$, and let Γ be its circumcircle. Let E and F be the midpoints of the sides AC and AB, respectively. The circumcircle of triangle CEF intersects Γ at X and C, with $X \neq C$. The line BX and the line tangent to Γ at A intersect at Y. Let P be the point on segment AB such that $YP = YA$, with $P \neq A$, and let Q be the point where AB intersects the line parallel to BC passing through Y. Show that F is the midpoint of PQ.

3. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers and let $b_1, b_2, b_3, \ldots$ be the sequence of real numbers given by

$$b_n = \frac{a_1 \cdot a_2 \ldots a_n}{a_1 + a_2 + \ldots + a_n}, \qquad \text{for } n \geq 1.$$

Show that, if for every one million consecutive terms of the sequence $b_1, b_2, b_3, \ldots$ there is at least one integer, then there is some k such that $b_k > 2021^{2021}$.

4. Let a, b, c, x, y, z be real numbers such that

$$a^2+x^2 = b^2+y^2 = c^2+z^2 = (a+b)^2+(x+y)^2 = (b+c)^2+(y+z)^2 = (c+a)^2+(z+x)^2.$$

Show that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$.

5. For a finite set C of integers, we define $S(C)$ to be the sum of the elements of C. Find two nonempty sets A and B, whose intersection is empty and whose union is the set $\{1, 2, \ldots, 2021\}$, such that the product $S(A)S(B)$ is a perfect square.

6. Consider a regular polygon with n sides, $n \geq 4$, and let V be a subset of r vertices of the polygon. Show that if $r(r-3) \geq n$ then there exist at least two congruent triangles whose vertices are in V.

Solutions

The 36th edition of the Iberoamerican Mathematical Olympiad took place in October 2021 in Costa Rica. Due to the Covid pandemic, the 91 participants cannot gather together and the competition followed an online format. It was developed in two consecutive days, and contestants had to solve 3 problems each day in a maximum time of four hours and a half. Each problem was graded with an integer mark between 0 and 7 points, so the maximum possible score was 42 points, achieved this year by three students. According to the usual standards, at most half of the participants can get a medal, and then these are awarded in the proportion 1:2:3 for gold, silver and bronze, respectively.
The Spanish team made a very good performance, achieving two silver medals (Leonardo Costa and Àlex Rodríguez) and two bronze medals (Bernat Pagès and Miguel Valdivieso). The chief of the delegation was Óscar Rivero and the deputy leader was Marc Felipe.
We present now the problems of the competition, and include the solutions given to them by our team. In all the cases, the solutions follow the ideas presented by the contestants, but we have done some little modifications to ease the exposition.

Problem 1. Let $P = \{p_1, p_2, \ldots, p_{10}\}$ be a set of 10 different prime numbers and let A be the set of all integers greater than 1 such that their prime factorizations contain only primes in P. Each element in A is colored in the following way:

(a) each element in P has a distinct color,

(b) if m, $n \in A$, then mn has the same color as either m or n,

(c) for each pair of distinct colors $\mathcal{R}$ and $\mathcal{S}$, there are no j, k, m, $n \in A$ (not necessarily distinct), with j, k colored $\mathcal{R}$ and m, n colored $\mathcal{S}$, such that both j divides m and n divides k.

Show that there is some prime in P such that all of its multiples in A have the same color.

Solution by Leonardo Costa.
First of all, by mathematical induction, all the powers of $p_1, p_2, \ldots, p_{10}$ must be painted using one of the colors of $p_1, p_2, \ldots, p_{10}$. We call those colors $1, 2, \ldots, 10$. Since any number of A is the product of powers of p_i, for $i \in \{1, \ldots, 10\}$, every number in A is of one of the colors $1, 2, \ldots, 9, 10$.
Consider the directed graph with vertices $1, \ldots, 10$ where i is connected with j (in this order) if there exists a vertex a of color i and a vertex b of color j such that $a \mid b$. Note that by condition (c), if i is connected with j, then j is not connected with i. Suppose that for any $i \in \{1, \ldots, 10\}$ there exists j such that i is connected to j. Then we may consider for any vertex a another node b such that a is connected with b, and this process can be successively iterated. Since the graph is directed, in some moment we will come back to a vertex we have already visited and we will have a cycle.

Let $x_1, x_2, \ldots, x_k$ be the vertices of the cycle. By simplicity, let $C_i = \{x \in A \mid x \text{ is of color } i\}$. Since x_1 is connected to x_2, $p_{x_1}p_{x_2} \in C_{x_2}$ (if it were in C_{x_1}, $p_{x_2} | p_{x_1}p_{x_2}$ and hence x_2 would be connected to x_1). Analogously, $p_{x_i}p_{x_{i+1}} \in C_{x_{i+1}}$. But then $p_{x_1}p_{x_2}\cdots p_{x_k} \in C_{x_1} \cup C_{x_2} \cup \ldots \cup C_{x_k}$.
Hence, if $p_{x_1}p_{x_2}\cdots p_{x_k} \in C_{x_j}$, then x_{j+1} is connected to x_j, contradicting condition (c). Note that in the previous sentence indices are taken modulo k. Finally, since we had assumed that for all $i \in \{1, \ldots, 10\}$ there exists j such that i is connected with j, there exists $i \in \{1, \ldots, 10\}$ such that for all j, i is not connected with j. Hence, for that i, since $p_i \in C_i$, every multiple of p_i will be also of color i, as desired.

Problem 2. Consider an acute triangle ABC, with $AC > AB$, and let Γ be its circumcircle. Let E and F be the midpoints of the sides AC and AB, respectively. The circumcircle of triangle CEF intersects Γ at X and C, with $X \neq C$. The line BX and the line tangent to Γ at A intersect at Y. Let P be the point on segment AB such that $YP = YA$, with $P \neq A$, and let Q be the point where AB intersects the line parallel to BC passing through Y. Show that F is the midpoint of PQ.

Solution by Bernat Pagès. Let $R = AC \cap QY$. Moving angles,

$$\gamma = \angle BCA = \angle FEA = \angle QRA,$$

and similarly

$$\gamma = \angle BCA = \angle QAY = \angle QPY,$$

where we have used that $AY = PY$. Further,

$$\beta = \angle CBA = \angle RQA = \angle PQY,$$

while

$$\angle QYP = 180^\circ - \angle PQY - \angle QPY = 180^\circ - \beta - \gamma = \alpha.$$

Then, YPQ is similar to ABC. Since $\angle RAP = \angle RYP = \alpha$, $RAYP$ is cyclic. Further, YAR is similar to YQA (comparing angles), and this means $YA^2 = YQ \cdot YR$. Using powers of Y in ABC, $YA^2 = YX \cdot YB$. These two equations imply that $QRXB$ is cyclic.
Note now that $\angle RXY = 180^\circ - \angle RXB = 180^\circ - \angle RQB = \angle ABR = \beta$. Since $\angle RAY + \angle RXY = \alpha + \beta + \gamma = 180^\circ$, so $XRAY$ is cyclic. All this together implies that $RAYXP$ is cyclic.
We now claim that R, F and X are collinear. Indeed, we already know that $\angle RXB = 180^\circ - \beta$. Regarding $\angle FXB$, observe that

$$\angle FXB = \angle FXC + \angle DXB = \angle FEA + \angle CAB = \alpha + \gamma.$$

Then, $\angle RXB = \angle FXB$, and then R, F and X are collinear.
Finally, using power of a point in the circles $RQXB$ and $AYXPR$, we obtain that

$$FX \cdot FR = FQ \cdot FB = FP \cdot FA.$$

Hence, $FQ \cdot FB = FP \cdot FA$. Since $FB = FA$, we conclude that $FQ = FP$, as desired.

Problem 3. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers and let $b_1, b_2, b_3, \ldots$ be the sequence of real numbers given by

$$b_n = \frac{a_1 \cdot a_2 \ldots a_n}{a_1 + a_2 + \ldots + a_n}, \qquad \text{for } n \geq 1.$$

Show that, if for every one million consecutive terms of the sequence $b_1, b_2, b_3, \ldots$ there is at least one integer, then there is some k such that $b_k > 2021^{2021}$.

Solution by Àlex Rodríguez. We proceed by contradiction, so assume that for any k, $b_k \leq 2021^{2021}$. Then, there exist infinitely many indexes i such that b_i is an integer between 1 and 2021^{2021}. By the pigeonhole principle, and since the distance between the integers is at most 10^6 there exist infinitely many pairs with $n < m$ such that $b_n = b_m$, $m - n < A := 10^6 \cdot 2021^{2021} + 1$. Then:

$$\frac{\prod_{i=1}^n a_i}{\sum_{i=1}^n} = \frac{\prod_{i=1}^m a_i}{\sum_{i=1}^m},$$

and from here we see that

$$\frac{\sum_{i=n+1}^m a_i}{\sum_{i=1}^n a_i} + 1 = \prod_{i=n+1}^m a_i.$$

In particular, the fraction in the left is an integer (and therefore, greater or equal than one). This implies that

$$\sum_{i=n+1}^m a_i \geq \sum_{i=1}^m a_i \geq \sum_{i=1}^n n.$$

In particular, we have seen that there exists a_i, with $n+1 \leq i \leq m$, and such that $a_i(m-n) \geq n$. By hypothesis,

$$a_i \cdot A > a_i(m-n) \geq n,$$

which implies that $a_i > \frac{n}{A}$, where n is arbitrarily large, but A is a constant. This means that a_n is not bounded (for any T there exists i such that $a_i > T$).
For our further use, note that $\frac{ab}{a+b} \geq 1$ if $a, b \geq 2$. Let us study now the growth of b_n:

$$b_{n+1} = \frac{\prod_{i=1}^n a_i \cdot a_{n+1}}{\sum_{i=1}^n a_i + a_{n+1}} = b_n\Big(\frac{a_{n+1} \cdot \sum_{i=1}^n a_i}{\sum_{i=1}^n a_i + a_{n+1}}\Big).$$

We may assume that $\sum_{i=1}^n a_i \geq 2$ for all $n \geq 2$, so $b_{n+1} \geq b_n$ if and only if $a_{n+1} \geq 1$ for all $n \geq 2$. We study now the minimum of b_n after one point. Let $t \in [3, 10^6 + 3]$ be an integer such that b_t is an integer (then $b_t \geq 1$). The unique option for the sequence (b_n) to decrease at that point is that $a_n = 1$, which can only happen 10^6 times in a row from t until b_n is an integer. The worst case scenario occurs when $a_n = 1$ for all $n = t+1, t+2, \ldots, t+10^6$. Then, since $b_t \geq 1$,

$$\prod_{i=1}^t a_i \geq \sum_{i=1}^t a_i \geq 3,$$

because $t \geq 3$. Therefore,

$$b_{t+10^6} = \frac{\prod_{i=1}^t a_i \prod_{i=t+1}^{t+10^6} a_i}{\sum_{i=1}^t a_i + \sum_{i=t+1}^{t+10^6} a_i} = \frac{\prod_{i=1}^t a_i}{\sum_{i=1}^t a_i + 10^6} \tag{4}$$

$$\geq 1 - \frac{10^6}{\sum_{i=1}^t a_i + 10^6} \geq 1 - \frac{10^6}{3 + 10^6} = B > 0. \tag{5}$$

This means we have found a lower, $b_n \geq B$, where B is a positive constant. Finally, consider the result which asserts that a_n is unbounded, and we add it to the relation

between b_{n+1} and b_n. Let k be an integer such that $a_k > T$, and sufficiently large such that $\sum_{i=1}^{k-1} a_i \geq \frac{2021^{2021}}{B}$.
Let $S = \sum_{i=1}^{k-1} a_i$. Then, $\prod_{i=1}^{k-1} a_i \geq SB$,

$$2021^{2021} \geq b_k = \frac{\prod_{i=1}^{k-1} a_i \cdot a_k}{S + a_k} \geq \frac{SBa_k}{S + a_k} \geq \frac{2021^{2022} a_k}{\frac{2021^{2022}}{B} + a_k},$$

where in the last inequality we have used that $\frac{x}{x+\alpha}$ is an increasing function for any $\alpha \geq 1$. Hence, if $C = \frac{2021^{2021} \cdot 2021^{2022}}{B}$,

$$C + 2021^{2021} a_k \geq 2021^{2022} a_k,$$

or what it is the same,

$$C \geq 2021^{2021} \cdot (2021 - 1) \cdot a_k,$$

contradicting the fact that (a_k) is unbounded.

Problem 4. Let a, b, c, x, y, z be real numbers such that

$$a^2 + x^2 = b^2 + y^2 = c^2 + z^2 = (a+b)^2 + (x+y)^2 = (b+c)^2 + (y+z)^2 = (c+a)^2 + (z+x)^2.$$

Show that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$.

Solution by Àlex Rodrǵuez. Write k for the common value of $a^2 + x^2$, $b^2 + y^2$, and so on. Then, adding the first three equations,

$$3k = a^2 + x^2 + b^2 + y^2 + c^2 + z^2.$$

Similarly, multiplying by two and subtracting the last three equations,

$$\begin{aligned} 3k &= -2(ab + bc + ca + xy + yz + zx) \\ &= -(a + b + c)^2 - (x + y + z)^2 + a^2 + b^2 + c^2 + x^2 + y^2 + z^2 \\ &= -(a + b + c)^2 - (x + y + z)^2 + 3k. \end{aligned}$$

Then,

$$a + b + c = x + y + z = 0.$$

By homogeneity, we may assume that $a^2 + x^2 = b^2 + y^2 = c^2 + z^2 = 1$ (the case where they are zero is trivial). Then,

$$a = \cos\alpha,\ x = \sin\alpha,\ b = \cos\beta,\ y = \sin\beta,\ c = \cos\gamma,\ z = \sin\gamma,$$

for some α, β, $\gamma \in \mathbb{R}$. Define complex numbers

$$z_1 = a + ix = e^{i\alpha}, \quad z_2 = b + iy = e^{i\beta}, \quad z_3 = c + iz = e^{i\gamma}.$$

Observe that $z_1 + z_2 + z_3 = 0$. Multiplying by $e^{-i\alpha}$, we obtain

$$1 + e^{i(\beta-\alpha)} + e^{i(\gamma-\alpha)}.$$

In particular, this means that $e^{i(\beta-\alpha)}$ and $e^{i(\gamma-\alpha)}$ are conjugate numbers with real part $-1/2$. In particular, z_1, z_2, z_3 give rise to an equilateral triangle over the unit circle. Then, we may assume without loss of generality that $\beta = \alpha + 2\pi/3$ and $\gamma = \alpha + 4\pi/3$. Further,

$$z_1 \cdot z_2 = e^{i\alpha} z_2, \quad z_1 \cdot z_3 = e^{i\alpha} z_3, \quad z_2 \cdot z_3 = e^{i\alpha} \cdot z_1,$$

and therefore

$$z_1z_2 + z_1z_3 + z_2z_3 = e^{i\alpha}(z_1 + z_2 + z_3) = 0.$$

This means that

$$z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1) = 0.$$

In particular, if $\Re(w)$ stands for the real part of w,

$$0 = \Re(z_1^2 + z_2^2 + z_3^2) = \cos^2\alpha - \sin^2\alpha + \cos^2\beta - \sin^2\beta + \cos^2\gamma - \sin^2\gamma.$$

We conclude that

$$a^2 + b^2 + c^2 = x^2 + y^2 + z^2,$$

as desired.

Problem 5. For a finite set C of integers, we define $S(C)$ to be the sum of the elements of C. Find two nonempty sets A and B, whose intersection is empty and whose union is the set $\{1, 2, \ldots, 2021\}$, such that the product $S(A)S(B)$ is a perfect square.

Solution by Miguel Valdivieso. Let $D = \{n \in \mathbb{N} \mid 1 \le n \le 2021\}$ and consider

$$A := \{n \in \mathbb{N} \mid 1 \le n \le 242\} \cup \{n \in \mathbb{N} \mid 1779 \le n \le 2021\}, \quad B = D \backslash A.$$

We will show that A and B satisfy the conditions of the statement. Indeed, using the definition of B, it is immediate showing that the intersection is the empty set and the union is D. Further, we can check that

$$S(A) = \sum_{i=1}^{242} i + \sum_{i=1779}^{2021} i = 243 \cdot 2021.$$

Moreover, since $S(D) = 2021 \cdot 2011$, we have that

$$S(B) = S(D) - S(A) = 2021 \cdot (1011 - 243)2021 \cdot 768.$$

We deduce that

$$S(A) \cdot S(B) = 2021^2 \cdot (243 \cdot 768) = (2^4 \cdot 3^3 \cdot 2021)^2,$$

as desired.

Remark. The key point is noting that $2021 \cdot 1011 = 43 \cdot 47 \cdot 3 \cdot 337$ must have a divisor of the form $4k + 1$, that is, that may be written as a sum of two squares. The only divisor of that form is $337 = 81 + 256$. From here, it is straightforward checking that any decomposition must be of the form $S(A) = 81 \cdot 3 \cdot 2021$, $S(B) = 256 \cdot 3 \cdot 2021$.

Problem 6. Consider a regular polygon with n sides, $n \ge 4$, and let V be a subset of r vertices of the polygon. Show that if $r(r - 3) \ge n$ then there exist at least two congruent triangles whose vertices are in V.

Solution by Gustavo Ochoa, proposer of the problem (it follows the ideas of the solution written by Leonardo Costa during the contest). The distance between two vertices of V, A and B, will be measured as the number of sizes needed to go from A to B using the shortest path. We will denote it as $d(A, B)$.

If among the r vertices of V there are four of them, A, B, C and D such that $d(A,B) = d(C,D)$, then a trivial check shows that there are two congruent triangles. Conversely, it is clear that if there are two congruent triangles, then there are four vertices belonging to those two triangles, A, B, C and D, such that $d(A,B) = d(C,D)$. Hence, it is enough with showing that if $r(r-3) \geq n$, then in V there are four distinct vertices, A, B, C and D, such that $d(A,B) = d(C,D)$.
Fixing a vertex $A \in V$, the possible distances to the other vertices of the set belong to $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Further, fixing a distance d, there are at most two vertices at a distance d of A. Note that if there are four vertices of V, B_1, C_1, B_2 and C_2, such that

$$d(A, B_1) = d(A, B_2), \quad d(A, C_1) = d(A, C_2),$$

then the triangles AB_1C_1 and AB_2C_2 are congruent (exchanging C_1 and C_2 if needed).
From now on, let us suppose that there are no two congruent triangles with vertices in V, and let us conclude that $r(r-3) < n$. When considering the distances from a vertex $A \in V$ to the remaining $r-1$ vertices in V, only one of those $r-1$ distances may be repeated. Hence, the cardinal of the set of distances from A must be greater or equal than $r-2$.
The number of possible distances among vertices of V (allowing repetition of distances which share a vertex, but counting just once $d(A,B) = d(B,A)$) is $\frac{r(r-1)}{2}$. As we have already discussed, among those sharing a vertex only one distance may be repeated. Further, if d is the repeated distance for vertex A and B we can derive a contradiction: in particular, assume that there exist C_1, C_2, D_1 and D_2 such that $d(A, C_1) = d(A, C_2) = d(B, D_1) = d(B, D_2)$, then the triangles AC_1C_2 and BD_1D_2 are congruent.
The number of possible distances is then $\lfloor n/2 \rfloor$. Since at most r of them are repeated, the number of possible distances (allowing repetitions) is $\lfloor n/2 \rfloor + r$. Therefore,

$$\frac{r(r-1)}{2} \leq \lfloor \frac{n}{2} \rfloor + r.$$

If n is odd, this directly implies that $r(r-3) < n$ and we are done. If n is even, it could happen that $r(r-3) = n$ without two congruent triangles. However, this is not possible. To achieve the equality, one of the distances must be $n/2$ (a diameter AB). In this case, to have equality, there must be two vertices of V, say C and D, at the same distance from A. Hence, C and D are also at the same distance from B, and so the triangles ACB and ADB are congruent.
Comment. There is a limit case where n is a multiple of three and there are three vertices A, B and C with $d(A,B) = d(B,C) = d(C,A) = n/3$. One can easily reach a contradiction in this situation.

RSME